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Lexicographic and sequential equilibrium problems

M. Bianchi · I. V. Konnov · R. Pini

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Abstract The aim of this work is to analyze lexicographic equilibrium problems on a topological Hausdorff vector space X, and their relationship with some other vector equilibrium problems. Existence results for the tangled lexicographic problem are proved via the study of a related sequential problem. This approach was already followed by the same authors in the case of variational inequalities.

Keywords Lexicographic equilibrium problem · Sequential equilibrium problem · Generalized monotonicity

1 Introduction

The formulation of equilibrium problems is quite general and makes it a versatile tool for the investigation of various problems arising in pure and applied mathematics, physics, economics and operation research. By a scalar equilibrium problem (EP) we understand the problem of finding $\overline{x} \in U$ such that $\phi(\overline{x}, y) \geq 0$, for every $y \in U$, where U is a given set, and $\phi : U \times U \to \mathbb{R}$. Even though many problems in nonlinear analysis can be regarded as

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M. Bianchi
Dipartimento di Discipline Matematiche, Finanza Matematica ed Econometria, Università Cattolica, Largo Gemelli, 1, 20123 Milano, Italy
e-mail: monica.bianchi@unicatt.it

I. V. Konnov (✉)
Department of System Analysis and Information Technologies, Kazan University, ul. Kremlevskaya, 18, 420008 Kazan, Russia
e-mail: konn-igor@yandex.ru; Igor.Konnov@ksu.ru

R. Pini
Dipartimento di Metodi Quantitativi per le Scienze Economiche e Aziendali, Università Milano-Bicocca, Via Bicocca degli Arcimboldi, 8, 20126 Milano, Italy
e-mail: rita.pini@unimib.it

particular cases of scalar EP (see, for instance, [1]), the quite recent investigation of vector problems, such as vector optimization and vector variational inequalities, led to the formulation of vector equilibrium problems that are based on cone orders, among them the Pareto one. However, from the theory of vector optimization, it is well known that the set of cone optimal solution points is somehow too large, so that one needs alternative approaches to refine it. One possibility is to use the lexicographic order, which has been recently investigated in connection with its applications in optimization and decision making theory (see [2] and the references therein).

This paper is devoted to lexicographic equilibrium problems and their relationship with some other vector equilibrium problems, in particular, sequential equilibrium problems. This approach was followed by the same authors in the case of variational inequalities. Indeed, in [3], they investigated equivalence properties between various kinds of lexicographic problems and sequential ones. They provided results for the tangled lexicographic problem via the study of the sequential problem, that admits simpler conditions for existence theorems.

Our aim is to analyze a lexicographic equilibrium problem defined via two bifunctions on a topological Hausdorff vector space X. In Sect. 2 different vector equilibrium problems are introduced and their relationships are investigated via generalized concepts of monotonicity. In Sect. 3, the existence of solutions for LEP is proved in case the lexicographic problem coincides with or contains the sequential one.

2 Different kinds of equilibrium problems

Let $\Phi = (\phi_1, \phi_2) : U \times U \to \mathbb{R}^2$, where U is a closed convex subset of a topological Hausdorff vector space X. We assume in the sequel that each ϕ_i is an *equilibrium bifunction*, i.e., $\phi_i(t,t) = 0$ for every $t \in U$. For greater convenience, given $K \subseteq U$, we denote by $\mathrm{EP}_i(K)$ $(i = 1, 2)$ the scalar equilibrium problem: find $\overline{x} \in K$ such that

$$\phi_i(\overline{x}, y) \geq 0, \quad \forall y \in K.$$

We denote by $S_{\mathrm{EP}_i}(K)$ the solution set of $\mathrm{EP}_i(K)$. A solution $\overline{x} \in S_{\mathrm{EP}_i}(K)$ will be called *strict* if $\phi_i(\overline{x}, y) > 0$, for every $y \in K,\ y \neq \overline{x}$. Notice that, if ϕ_i is *pseudomonotone*, that is, for every $x, y \in U$,

$$\phi_i(x, y) > 0 \Longrightarrow \phi_i(y, x) < 0,$$

then any strict solution is also the unique solution of $\mathrm{EP}_i(K)$.

Let us now consider the following vector equilibrium problems in the space $\mathbb{R}^2$ endowed with the order relation induced by the Paretian cone:

(*weak equilibrium problem*) find $\overline{x} \in U$ such that

$$\Phi(\overline{x}, y) \nless \underline{0}, \quad \forall y \in U \tag{W-VEP}$$

(*strong equilibrium problem*) find $\overline{x} \in U$ such that

$$\Phi(\overline{x}, y) \geq \underline{0}, \quad \forall y \in U \tag{S-VEP}$$

(*sequential equilibrium problem*) find $\overline{x} \in U$ such that

$$\begin{cases} \phi_1(\overline{x}, y) \geq 0, & \forall y \in U \\ \phi_2(\overline{x}, y) \geq 0, & \forall y \in U_1^* = S_{\mathrm{EP}_1}(U) \end{cases} \tag{SQ-VEP}$$

(*lexicographic equilibrium problem*) find $\overline{x} \in U$ such that

$$\Phi(\overline{x}, y) \geq_\ell \underline{0}, \quad \forall y \in U \tag{LEP}$$

where $a \geq_\ell \underline{0}$ means that either $a_1 > 0$ or $a_1 = 0$ and $a_2 \geq 0$

The set of the solutions of the previous problems in U will be denoted by

$$S_{\text{W-VEP}}(U), \quad S_{\text{S-VEP}}(U), \quad S_{\text{SQ-VEP}}(U), \quad S_{\text{LEP}}(U),$$

respectively. It is trivial that

$$S_{\text{S-VEP}}(U) = S_{\text{EP}_1}(U) \cap S_{\text{EP}_2}(U) \subseteq S_{\text{LEP}}(U) \subseteq S_{\text{W-VEP}}(U).$$

Moreover,

$$S_{\text{S-VEP}}(U) \subseteq S_{\text{SQ-VEP}}(U) \subseteq S_{\text{W-VEP}}(U).$$

On the other hand, it is not clear, a priori, what is the relationship between $S_{\text{LEP}}(U)$ and $S_{\text{SQ-VEP}}(U)$.

It is worth noticing that LEP can be written in the following equivalent way: find $\overline{x} \in U$ such that

$$\begin{cases} \phi_1(\overline{x}, y) \geq 0, & \forall y \in U, \\ \phi_2(\overline{x}, z) \geq 0, & \forall z \in \mathcal{Z}(\overline{x}); \end{cases} \tag{1}$$

where

$$\mathcal{Z} : S_{\text{EP}_1}(U) \to 2^U, \quad \mathcal{Z}(x) = \{y \in U : \ \phi_1(x, y) = 0\}.$$

From (1), if $\overline{x}$ is a strict solution, then $\mathcal{Z}(\overline{x}) = \{\overline{x}\}$; in particular, the set of the strict solutions of $\text{EP}_1(U)$ is contained in $S_{\text{LEP}}(U)$. If $\overline{x} \in \text{EP}_1(U)$, but it is not a strict solution, then clearly

$$\mathcal{Z}(\overline{x}) = \{y \in U : \ \phi_1(\overline{x}, y) \leq 0\},$$

hence $\mathcal{Z}(\overline{x})$ is nonempty, compact and convex if we assume that $\phi_1(x, \cdot)$ is lower semicontinuous and quasiconvex, for every $x \in U$.

Next results show that the bifunction ϕ_1 and its (generalized) monotonicity properties play an interesting role in the comparison between LEP and SQ-VEP.

Proposition 1 *If ϕ_1 is pseudomonotone, then $S_{LEP}(U) \subseteq S_{SQ\text{-}VEP}(U)$.*

Proof We show that $S_{\text{EP}_1}(U) \subseteq \mathcal{Z}(x)$, for every $x \in S_{\text{EP}_1}(U)$. Indeed, fix $x^* \in S_{\text{EP}_1}(U)$, and suppose, by contradiction, that there exists $x \in S_{\text{EP}_1}(U)$ such that $x^* \notin \mathcal{Z}(x)$. Then,

$$\phi_1(x^*, x) \geq 0, \qquad \phi_1(x, x^*) > 0.$$

By the pseudomonotonicity, we get

$$\phi_1(x, x^*) \leq 0,$$

a contradiction. From (1), the result follows. □

It is worthwhile noticing that the condition $\mathcal{Z}(x) \subseteq S_{\text{EP}_1}(U)$ for every $x \in U$ is sufficient to entail the reverse inclusion $S_{\text{SQ-VEP}}(U) \subseteq S_{\text{LEP}}(U)$. In order to provide a result for the equivalence of the two solution sets, we need the following

Definition 1 A bifunction $\Psi : X \times X \to \mathbb{R}$ is said to be

(a) *skew-symmetric*, if for each pair of points $x, y \in X$, the following equality holds:

$$\Psi(x, y) + \Psi(y, x) = 0;$$

(b) *pseudo-symmetric*, if for each pair of points $x, y \in X$, the following implication holds:

$$\Psi(x, y) = 0 \Rightarrow \Psi(y, x) = 0;$$

(c) *pseudomonotone*$_*$, if it is pseudomonotone, and, for each pair of points $x, y \in X$, if

$$\Psi(x, y) = 0, \qquad \Psi(y, x) = 0,$$

then, for every $w \in X$,

$$\Psi(x, w) \gtreqless 0 \Longrightarrow \Psi(y, w) \gtreqless 0;$$

(d) *cyclically monotone* if for every $x_1, x_2, \ldots, x_n \in X$, $n \in \mathbb{N}$, the following implication holds:

$$\sum_{i=1}^{n} \Psi(x_i, x_{i+1}) \leq 0,$$

where $x_{n+1} = x_1$;

(e) *cyclically pseudomonotone* if for every $x_1, x_2, \ldots, x_n \in X$, $n \in \mathbb{N}$, the following implication holds:

$$\exists i \in \{1, 2, \ldots, n\},\ \Psi(x_i, x_{i+1}) > 0 \Rightarrow \exists j \in \{1, 2, \ldots, n\},\ \Psi(x_j, x_{j+1}) < 0,$$

where $x_{n+1} = x_1$;

(f) *pseudoaffine*, if both Ψ and $-\Psi$ are pseudomonotone;

(g) *pseudo-symmetric*$_*$, if Ψ is pseudo-symmetric and pseudomonotone$_*$.

Concept (d) extends the well known cyclic monotonicity for mappings. Cyclical monotonicity of a bifunction Ψ can be characterized via the existence of a function $\phi : X \to \mathbb{R}$ such that

$$\Psi(x, y) \leq \phi(y) - \phi(x),$$

for every $x, y \in X$ (see, for instance, [4]).

Concept (e) extends the one introduced in [5] for mappings. Since the implication holds for any integer n and, in particular, for $n = 2$, each cyclically pseudomonotone bifunction is clearly pseudomonotone. Note that each pseudoaffine bifunction is pseudo-symmetric and any skew-symmetric bifunction is pseudo-symmetric and pseudomonotone. Next lemma shows that also cyclical pseudomonotonicity and pseudo-symmetry$_*$ are related.

Lemma 1 *If a bifunction $\Psi : U \times U \to \mathbb{R}$ is cyclically pseudomonotone and pseudoaffine, then Ψ is pseudo-symmetric$_*$.*

Proof Under the above assumptions Ψ is clearly pseudo-symmetric and pseudomonotone. Fix a pair of points $x, y \in X$ such that $\Psi(x, y) = 0$, then $\Psi(y, x) = 0$. Take any point $w \in X$.

Case 1. If $\Psi(x, w) > 0$, then taking the triplet $x_1 = y, x_2 = x$ and $x_3 = w$ gives $\Psi(w, y) < 0$, hence $\Psi(y, w) > 0$ by pseudoaffinity.

Case 2. If $\Psi(x, w) < 0$, then $\Psi(w, x) > 0$, and taking the triplet $x_1 = w$, $x_2 = x$ and $x_3 = y$ gives $\Psi(y, w) < 0$.

Case 3. If $\Psi(x, w) = 0$, then $\Psi(y, w) \neq 0$ leads to a contradiction due to Cases 1 and 2 with replacing x and y. □

Notice that the same conclusion holds if $-\Psi$ is assumed to be cyclically pseudomonotone and pseudoaffine.

Assuming that ϕ_1 satisfies some of the definitions given above, we can show the equivalence between the sets $S_{\text{SQ-VEP}}(U)$ and $S_{\text{LEP}}(U)$.

Proposition 2 *If ϕ_1 is pseudo-symmetric$_*$, then $S_{LEP}(U) = S_{SQ\text{-}VEP}(U)$.*

Proof From Proposition 1, since ϕ_1 is in particular pseudomonotone, we need to show that $S_{\text{SQ-VEP}}(U) \subseteq S_{\text{LEP}}(U)$. To this purpose, we prove that $\mathcal{Z}(\overline{x}) \subseteq S_{\text{EP}_1}(U)$, for every $\overline{x} \in S_{\text{EP}_1}(U)$. Fix $\overline{y} \in \mathcal{Z}(\overline{x})$. If $\overline{y} \in \mathcal{Z}(\overline{x})$, then $\phi_1(\overline{x}, \overline{y}) = 0$; it follows by the assumptions that $\phi_1(\overline{y}, \overline{x}) = 0$. By the definition of pseudomonotonicity$_*$ the inequality

$$\phi_1(\overline{x}, w) \geq 0, \quad \forall w \in U$$

implies that

$$\phi_1(\overline{y}, w) \geq 0, \quad \forall w \in U.$$

Therefore, $\overline{y} \in S_{\text{EP}_1}(U)$. □

From Lemma 1 and Proposition 2 we easily get the following

Corollary 1 *If ϕ_1 is cyclically pseudomonotone and $-\phi_1$ is pseudomonotone, or, $-\phi_1$ is cyclically pseudomonotone and ϕ_1 is pseudomonotone, then $S_{LEP}(U) = S_{SQ\text{-}VEP}(U)$.*

In the particular case where $\phi_1(x, y) = \langle A(x), y - x\rangle$, with $A : U \to X^*$, another set of conditions entails the equivalence of the solution sets of LEP and SQ-VEP. Indeed, under the assumptions of the following corollary, trivial computations show that any point in $\mathcal{Z}(x)$ is a solution of EP_1.

Proposition 3 *Assume that A satisfies the following assumptions:*

(a) *A is pseudoaffine, i.e., both A and $-A$ are pseudomonotone;*
(b) *A is pseudomonotone$_*$ (according to [6, Sect. 2.2]), i.e., for each pair of points $u, v \in U$, if*

$$\langle A(u), v - u\rangle = 0, \qquad \langle A(v), v - u\rangle = 0,$$

then there exists $\mu > 0$ such that $A(u) = \mu A(v)$.

Then $S_{LEP}(U) = S_{SQ\text{-}VEP}(U)$.

Now we intend to investigate conditions which provide only the inclusion $S_{\text{SQ-VEP}}(U) \subseteq S_{\text{LEP}}(U)$. More precisely, we will utilize distance type properties of the bifunction ϕ_1. In [7,8] and references therein, existence results for equilibrium problems via the Ekeland variational principle were proved under the *triangular property*

$$\Psi(x, y) \leq \Psi(x, z) + \Psi(z, y) \quad \forall\, x, y, z \in U.$$

Notice that any bifunction satisfying this property is the opposite of a cyclically monotone bifunction (see [7]).

Several examples of bifunctions satisfying the triangular property are given e.g. in [8]. Nevertheless, in the particular case where $\Psi(x, y) = \langle A(x), y - x\rangle$ with $A : U \to X^*$, the triangular property becomes

$$\langle A(z) - A(x), y - z\rangle \geq 0 \quad \forall\, x, y, z \in U,$$

which seems rather restrictive. For this reason, we now introduce the essentially weakened property.

Definition 2 A bifunction $\Psi : X \times X \to \mathbb{R}$ is said to be a *weak pseudo-distance*, if for each triplet $x, y, z \in X$, there exist numbers $\alpha > 0$ and $\beta > 0$, which may depend on x, y, and z, such that the following implication holds:

$$\Psi(x, y) \geq 0 \implies \alpha\Psi(x, z) + \beta\Psi(z, y) \geq 0.$$

Proposition 4 *If ϕ_1 is a weak pseudo-distance, then $S_{SQ\text{-}VEP}(U) \subseteq S_{LEP}(U)$.*

Proof In fact, it suffices to prove that $\mathcal{Z}(\overline{x}) \subseteq S_{\mathrm{EP}_1}(U)$ for every $\overline{x} \in S_{\mathrm{EP}_1}(U)$. Fix $\overline{z} \in \mathcal{Z}(\overline{x})$, then $\phi_1(\overline{x}, \overline{z}) = 0$ and, for each $y \in K$, it holds that

$$\phi_1(\overline{x}, y) \geq 0.$$

It follows by the assumptions that

$$\phi_1(\overline{z}, y) \geq 0.$$

Therefore, $\overline{z} \in S_{\mathrm{EP}_1}(U)$. □

3 Existence results

The direct investigation of a lexicographic equilibrium problem is not an easy task. Quite strong assumptions entail the nonemptiness of $S_{\text{S-VEP}}(U)$, and, as a matter of fact, every LEP is solved by any solution of the S-VEP. In [9], for instance, the authors proved existence results for the S-VEP where a monotone family of bifunctions is involved.

In this section, we are interested in finding weaker conditions that ensure the nonemptiness of $S_{\text{LEP}}(U)$. By the results of the end of Sect. 2, a possible approach for the existence of solutions of LEP can be provided via the analysis of the solutions of SQ-VEP, under the assumptions of Propositions 2–4.

Several authors studied sequential problems, especially exploiting its relation with regularization methods; see, for instance, [10] and the references therein. Chadli et al. [11] obtained conditions for the existence of solutions of a sequential equilibrium problem via a viscosity argument under quite strong conditions.

In this section, we assume that U is a nonempty compact convex subset of a topological Hausdorff vector space X, unless otherwise stated. We utilize here this compactness assumption only for the sake of simplicity of exposition. In the existence results it can be replaced by the previous closedness of U and the corresponding coercivity condition; see e.g. [1,12,13] and references therein.

Observe that the set $S_{\text{SQ-VEP}}(U)$ is nonempty if $S_{\mathrm{EP}_1}(U)$ is nonempty, convex, and compact and $S_{\mathrm{EP}_2}(U)$ is nonempty. Therefore, we should utilize suitable existence results for scalar equilibrium problems.

Let us consider the scalar EP: find $\overline{x} \in U$ such that

$$\Psi(\overline{x}, y) \geq 0 \qquad \forall y \in U, \tag{2}$$

where $\Psi : U \times U \to \mathbb{R}$ is an equilibrium bifunction and denote by U^* its solution set. We also need some additional definitions. First we recall some generalized convexity and continuity properties of scalar functions.

Definition 3 A function $f : X \to \mathbb{R}$ is said to be

(a) *quasiconvex*, if for each pair of points $x', x'' \in X$ and for all $\alpha \in [0, 1]$, we have

$$f(\alpha x' + (1 - \alpha)x'') \leq \max\{f(x'), f(x'')\};$$

(b) *semistrictly quasiconvex*, if for each pair of points $x', x'' \in X$ such that $f(x') \neq f(x'')$ and for all $\alpha \in (0, 1)$, we have

$$f(\alpha x' + (1 - \alpha)x'') < \max\{f(x'), f(x'')\};$$

(c) *lower semicontinuous*, if the lower level set

$$\text{lev}_{\leq\alpha} f = \{x \in X : f(x) \leq \alpha\}$$

is closed, for every $\alpha \in \mathbb{R}$;

(d) *lower hemicontinuous* if its restriction on the line segments of X is lower semicontinuous, i.e.,

$$\liminf_{t \downarrow 0^+} f(x + t(y - x)) \geq f(x), \quad \forall x \in X.$$

A classical result by Karamardian entails that every lower semicontinuous and semistrictly quasiconvex function is quasiconvex. Moreover, a function f is said to be, respectively, *quasiconcave, semistrictly quasiconcave, upper semicontinuous, upper hemicontinuous* if $-f$ is quasiconvex, semistrictly quasiconvex, lower semicontinuous, lower hemicontinuous.

Well-known results about existence of equilibria and properties of U^* are based on convexity and monotonicity assumptions on the bifunction Ψ. On this subject, we would like to recall one of the former results by Brézis et al. (see [14, Application 2]), where it is proved that U^* is nonempty, convex, and compact assuming the following set of conditions:

(A1) Ψ *is pseudomonotone,* $\Psi(x, \cdot)$ *is semistrictly quasiconvex and lower semicontinuous for each* $x \in U$, $\Psi(\cdot, y)$ *is upper hemicontinuous for each* $y \in U$.

Taking into account Proposition 2 and Lemma 1, we obtain the following existence result for SQ-VEP and LEP:

Theorem 1 *Assume that the bifunctions* ϕ_1 *and* ϕ_2 *satisfy the conditions in* **(A1)** *as* Ψ. *Then the set* $S_{SQ\text{-}VEP}(U)$ *is nonempty, convex, and compact. If, in addition,* ϕ_1 *satisfies one of the following assumptions:*

(a) ϕ_1 *is pseudo-symmetric*$_*$,
(b) $-\phi_1$ *is cyclically pseudomonotone,*
(c) ϕ_1 *is cyclically pseudomonotone and* $-\phi_1$ *is pseudomonotone;*

then $S_{LEP}(U) = S_{SQ\text{-}VEP}(U)$.

It is worthwhile noticing that conditions appearing in **(A1)** are quite strong; indeed, they provide the equivalence between EP and its dual problem defined as follows: find $\overline{y} \in U$ such that

$$\Psi(x, \overline{y}) \leq 0 \qquad \forall x \in U.$$

In particular, as a by-product, they give also the compactness and the convexity of the solution set.

To the aim of weakening some of the assumptions in **(A1)**, several results can be found in literature; most of them provide only existence and compactness results for EP. Let us first recall the following generalized monotonicity property:

Definition 4 A bifunction $\Psi : X \times X \to \mathbb{R}$ is said to be *properly quasimonotone* if, for every finite set $A \subset U$, and for every $x \in co(A)$, it holds that $\max_{y \in A} \Psi(x, y) \geq 0$.

In [15] some sufficient conditions implying proper quasimonotonicity of a bifunction are proved; we recall two of them:

(1) $\Psi(x, \cdot)$ is quasiconvex, for every $x \in X$;
(2) $\Psi(\cdot, y)$ quasiconcave, for every $y \in X$, and $-\Psi$ pseudomonotone.

The following set of conditions entail existence and compactness of U^* (for further weakenings see, also, [12]):

(A2) $\Psi(x, \cdot)$ *is properly quasimonotone for each* $x \in U$, $\Psi(\cdot, y)$ *is upper semicontinuous for each* $y \in U$.

Notice that the proper quasimonotonicity in **(A2)** can be replaced by one of the conditions (1) or (2).

In order to get also convexity of the equilibria, we need to assume, in addition,

(A3) $\Psi(\cdot, y)$ is *quasiconcave*.

Combining the previous conditions we get next result:

Theorem 2 *Assume that the bifunction* ϕ_1 *satisfies the conditions in* **(A1)***, or, otherwise,* **(A2)** *and* **(A3)***, and* ϕ_2 *satisfies the conditions in* **(A2)***. Then the set* $S_{SQ\text{-}VEP}(U)$ *is nonempty and compact. If, in addition,* ϕ_1 *is a weak pseudo-distance, then* $S_{SQ\text{-}VEP}(U) \subseteq S_{LEP}(U)$*. Otherwise, if* ϕ_1 *satisfies one of the following assumptions:*

(a) ϕ_1 *is pseudo-symmetric*$_*$,
(b) $-\phi_1$ *is cyclically pseudomonotone,*
(c) ϕ_1 *is cyclically pseudomonotone and* $-\phi_1$ *is pseudomonotone;*

then $S_{LEP}(U) = S_{SQ\text{-}VEP}(U)$.

In an Euclidean setting, another set of sufficient conditions for existence and compactness of the solutions involve distance type properties of the bifunction Ψ (see [7], and the references therein):

(A4) Ψ *possesses the triangle property,* $\Psi(x, \cdot)$ *is lower semicontinuous for each* $x \in U$, $\Psi(\cdot, y)$ *is upper semicontinuous for each* $y \in U$.

Notice that this kind of conditions does not require any convexity of the domain. Within this framework, we have the following result:

Theorem 3 *Assume that* U *is a nonempty compact subset of on Euclidean space* X*, the bifunctions* ϕ_1 *and* ϕ_2 *satisfy the conditions in* **(A4)** *as* Ψ*. Then the set* $S_{SQ\text{-}VEP}(U)$ *is nonempty and compact and* $S_{SQ\text{-}VEP}(U) \subseteq S_{LEP}(U)$.

Again, we can combine the assumptions on ϕ_1 and ϕ_2 from Theorems 2 and 3, if necessary, to obtain the existence results for SQ-VEP and LEP.

A class of functions ϕ satisfying the assumptions in (**A1**) can be built up as follows. Let $h : X \to \mathbb{R}$ be a semistrictly quasiconvex and lower semicontinuous function, and $g : \mathbb{R} \to \mathbb{R}$ be an increasing, lower semicontinuous function such that $g(0) = 0$. Define

$$\phi(x, y) = g(h(y) - h(x)) : X \times X \to \mathbb{R}.$$

It is a straightforward exercise to show that ϕ satisfies all the assumptions in (**A1**). Indeed:

(1) ϕ is pseudomonotone, since $g(h(y)-h(x)) > 0$ is equivalent to $h(y)-h(x) > 0$, implying that $h(x)-h(y) < 0$, and, by the assumptions, $g(h(x)-h(y)) < 0$ (it is worthwhile to notice that also $-\phi$ is pseudomonotone);
(2) $\phi(x,\cdot)$ is semistrictly quasiconvex for every $x \in X$, since $h(\cdot)-h(x)$ is semistrictly quasiconvex, and g is increasing (see [16], p. 154);
(3) $\phi(x,\cdot)$ is lower semicontinuous for every $x \in X$; indeed,

$$\text{lev}_{\leq\alpha}\phi(x,\cdot) = \{y \in X : h(y) \in h(x) + g^{-1}(-\infty,\alpha]\},$$

that is closed for every α, since h is lower semicontinuous and, by the assumptions on g, $g^{-1}(-\infty,\alpha] = (-\infty,\beta]$, for a suitable $\beta \in \mathbb{R}$.
(4) $\phi(\cdot,y)$ is upper hemicontinuous for every $y \in X$; indeed, in this case, $\phi(\cdot,y)$ is even upper semicontinuous.

Moreover, the function $\phi(x,y) = g(h(y)-h(x))$ is also pseudo-symmetric$_*$: it is trivial that $\phi(x,y) = 0$ implies $\phi(y,x) = 0$; furthermore, since $\phi(x,y) = 0$ if and only if $h(x) = h(y)$, we obtain immediately the pseudomonotone$_*$ assumption.

Another class of functions that satisfy assumptions in (**A1**) can be recovered by taking $\phi(x,y) = h(y-x)$, where $h : X \to \mathbb{R}$ is a semistrictly quasiconvex and lower semicontinuous function such that:

(a) $h(t) > 0$ implies $h(-t) < 0$;
(b) $h(0) = 0$.

Moreover, the function $-\phi$ is cyclically monotone (and, therefore, cyclically pseudomonotone) if there exists a function $k : X \to \mathbb{R}$ such that

$$h(y-x) \geq k(x) - k(y).$$

An interesting application of the previous results can be given in the setting of the theory of nontransitive consumer (see [17,18]).

We recall that the preference R of a nontransitive consumer can be expressed by a skew-symmetric representation function $r : \mathbb{R}^+_l \times \mathbb{R}^+_l \to \mathbb{R}$ such that

$$xRy \iff r(x,y) \geq 0.$$

Two commodity bundles x and y are indifferent (xIy) if and only if $r(x,y) = 0$, while x is strictly preferred to y (xPy) if and only if $r(x,y) > 0$. It is well known that I is an equivalence relation. Usual demands on the bifunction r are its concave–convexity, together with its continuity. The assumptions in (**A1**) are therefore fulfilled. Another reasonable assumption on r is the following: if xIy, then

$$xPw \iff yPw,$$

i.e., if $r(x,y) = 0$, then

$$r(x,w) > 0 \iff r(y,w) > 0.$$

In particular, the equivalence above entails that r is a pseudomonotone$_*$ bifunction.

Given the budget set $B(p,w) = \{x \in \mathbb{R}^+_l : xp \leq w\}$, x^* is an optimal demand if and only if x^* solves the equilibrium problem

$$r(x^*,x) \geq 0, \quad \forall x \in B(p,w).$$

One can argue that the results in Theorem 1 can be applied to multicriteria problems as well as to social choice.

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